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Stationary Diffusion over a Multidimensional Potential Barrier: A Generalization of Kramers' Formula

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We consider diffusion over a potential barrier for n degrees of freedom. Generalizing the procedure of Kramers, we find a quasistationary solution to the associated Fokker–Planck equation. This yields an expression for the diffusion current over the barrier and, finally, a simple and elegant generalization of Kramers' formula for the diffusion rate.

KEY WORDS: Fokker–Planck equation; stationary diffusion current.

1. INTRODUCTION

As early as 1940, Kramers[1] proposed that induced nuclear fission can be viewed as a diffusion process over a potential barrier. On the basis of this picture, he wrote down and solved the Fokker–Planck equation for one degree of freedom, the fission variable x, in a quasistationary approximation. This procedure yielded for the fission width Γ_f the following expression:

$$\Gamma_f = \frac{\hbar}{2\pi} \exp\left(\frac{-V_B}{kT} \right)\left(\frac{W}{|V|} \right)^{1/2} h \tag{1.1}$$

Here, $V_B > 0$ is the height of the fission barrier, T the nuclear temperature, k Boltzmann's constant, and $\hbar$ Planck's constant divided by 2π. The remaining factors are defined as follows.

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Near the top $x = x^0$ of the barrier, the potential $V(x)$ is written as

$$V(x) \cong \tfrac{1}{2} V(x - x^0)^2 \qquad \text{with} \quad V < 0 \tag{1.2}$$

Moreover, $V(x)$ has a local minimum at $x = 0$. This minimum defines the deformation of the nucleus before it undergoes fission. Near $x = 0$, we have

$$V(x) \cong \tfrac{1}{2} W x^2 - V_B \qquad \text{with} \quad W > 0 \tag{1.3}$$

The constant h is the positive root of the quadratic equation

$$Mh^2 + \beta h + V = 0 \tag{1.4}$$

Here, M is the mass (inertial parameter) associated with the fission variable x, and β is the friction constant. It is easy to show that

$$0 \leqslant h \leqslant (|V|/M)^{1/2} \tag{1.5}$$

and that h decreases monotonically with increasing β. Equation (1.1) yields an excellent approximation to the actual fission width as defined by the solution of the Fokker–Planck equation, except for very small values of β (which are physically unrealistic in the nuclear context). Kramers' work has subsequently also found application to other problems not connected to nuclear physics like the autoionization of molecules.(2)

In the present paper, we show that Kramer's method, and the resulting formula (1.1), can easily be extended to a quasistationary diffusion process involving several (n) degrees of freedom. Such a generalization is called for in the nuclear context where it is known that several degrees of freedom are required to adequately describe the shapes of fissioning nuclei.(3) It is conceivable that other problems to which the diffusion equation has been applied call for a similar extension. Several attempts have actually been made in this direction in the context of nuclear physics.(4,5) In the present paper, we carry the algebraic evaluation of Γ_f considerably beyond the stage attained in these papers.

The problem of diffusion over a barrier in several dimensions has a long history, and several general mathematical approaches towards this problem exist which are considerably more powerful than the techniques used below. (See Ref. 6 for a recent review.) We believe, however, that our result given in Section 5 below has the advantage of displaying explicitly the dependence of the decay rate on the relevant parameters—the shape of the potential landscape, the mass tensor, and the friction tensor—and that it is therefore useful for a comparison with experimental data.

In Section 2, we define the Fokker–Planck equation and generalize the quasistationary approximation of Kramers. The resulting eigenvalue equation is shown in Section 3 to have only one positive root. For this solution, the quasistationary current over the barrier is calculated in Section 4, and Section 5 contains a discussion of our results.

2. QUASISTATIONARY SOLUTION OF THE FOKKER–PLANCK EQUATION

We consider n degrees of freedom with coordinates $x_1, \ldots, x_n$ and velocities $u_1, \ldots, u_n$. The mass tensor M_{ij} and the friction tensor β_{ij} $(i, j = 1, \ldots, n)$ are assumed to be real, symmetric, and positive definite. We also assume these tensors to be independent of $x_1, \ldots, x_n$ although our procedure can be extended to allow for a smooth x-dependence of M and β. The potential $V(x_1, \ldots, x_n)$ has a local minimum at $x_1 = x_2 = \cdots = x_n = 0$. This minimum defines the shape of the nucleus before it undergoes fission. Near the minimum, we write

$$V(x_1, \ldots, x_n) \cong \tfrac{1}{2} W_{ij} x_i x_j - V_B \tag{2.1}$$

where we use the summation convention. The matrix W_{ij} is real, symmetric, and positive definite. We also assume that $V(x_1, \ldots, x_n)$ has a single saddle point at $x_i = x_i^0$. Near this point, we have

$$V(x_1, \ldots, x_n) \cong \tfrac{1}{2} V_{ij}\left(x_i - x_i^0\right) \cdot \left(x_j - x_j^0\right) \tag{2.2}$$

Comparison of Eqs. (2.1) and (2.2) shows that $V_B > 0$ defines the height of the fission barrier. To make sure that there exists only a single fission path connecting the minimum at $x_i = 0$ with the asymptotic domain, we assume that the real and symmetric matrix V_{ij} has one negative and $(n-1)$ positive eigenvalues. This assumption is crucial for all that follows.

The probability density $W(x_1, \ldots, x_n; u_1, \ldots, u_n; t)$ in phase space, a function of time t, obeys the following Fokker–Planck equation:

$$\frac{\partial W}{\partial t} + u_i \frac{\partial W}{\partial x_i} - (M^{-1})_{ij} \frac{\partial V}{\partial x_i} \frac{\partial W}{\partial u_j} = \frac{\partial}{\partial u_i}\left[(M^{-1})_{ij} \beta_{jk} u_k W\right] + \frac{\partial^2}{\partial u_i \partial u_j}(D_{ij} W) \tag{2.3}$$

The diffusion tensor D_{ij} is connected with the friction tensor β_{ij} through the generalized Einstein relation (fluctuation-dissipation theorem)

$$D_{ij} = KT(M^{-1})_{il} B_{lm} (M^{-1})_{mj} \tag{2.4}$$

The Fokker–Planck equation (2.3) describes nuclear fission as a diffusion process in an n-dimensional potential landscape. The diffusion is triggered by the coupling of the $x_1, \ldots, x_n$ to the other degrees of freedom of the system. This coupling is summarily described by the friction tensor β_{ij}. The other degrees of freedom act as a heat bath of temperature T. This description is adequate if we consider *induced* nuclear fission since the nucleonic degrees of freedom excited in a nuclear reaction induced by light projectiles equilibrate over a time scale short in comparison with typical fission times as given by the ratio of the mass over the nuclear friction constant. The use of the temperature concept is permitted as long as the nuclear excitation energy is substantially larger than the height V_B of the fission barrier since then the energy spent on climbing the barrier does not affect the nuclear temperature.

It is convenient to simplify Eq. (2.3) by a change of variables. We introduce the matrix $M^{1/2}$, avoiding a sign ambiguity by requiring the eigenvalues of $M^{1/2}$ to be all positive. We define ($i = 1, \ldots, n$)

$$\begin{aligned} y_i &= (M^{1/2})_{ij} x_j, \qquad s_i = (M^{1/2})_{ij} u_j \\ \gamma_{ij} &= (M^{-1/2})_{il} \beta_{lm} (M^{-1/2})_{mj} \end{aligned} \tag{2.5}$$

With these definitions and Eq. (2.4), Eq. (2.3) takes the form

$$\frac{\partial W}{\partial t} + s_i \frac{\partial W}{\partial y_i} - \frac{\partial V}{\partial y_i} \cdot \frac{\partial W}{\partial s_i} = \frac{\partial}{\partial s_i} (\gamma_{ij} s_j W) + kT\gamma_{ij} \frac{\partial^2}{\partial s_i \partial s_j} W \tag{2.6}$$

where W is now a function of $y_1, \ldots, y_n; s_1, \ldots, s_n; t$ and V a function of $y_1, \ldots, y_n$. In the new variables, the location of the saddle point is given by $(y_1^0, \ldots, y_n^0)$. Since β is positive definite, so is γ.

To describe the quasistationary diffusion process (the leakage of probability density from the local minimum at $y_i = 0$ over the saddle point to the asymptotic region), we generalize Kramers' original ansatz[1] for W as follows:

$$\begin{aligned} W(y_1, \ldots, y_n; s_1, \ldots, s_n) \\ = F(y_1, \ldots, y_n; s_1, \ldots, s_n) \\ \times \exp\left\{ -\frac{1}{2kT} \left[s_i s_i + 2V(y_1, \ldots, y_n) \right] \right\} \end{aligned} \tag{2.7}$$

The last factor represents the equilibrium phase space density and, taken by itself, could not describe any diffusion. This shortcoming is made up for by the function F which is chosen to be unity at $y_i = 0$, and zero asymptotically. Substitution of Eq. (2.7) into Eq. (2.6) yields for F the equation

$$s_i \frac{\partial F}{\partial y_i} - \frac{\partial V}{\partial y_i} \frac{\partial F}{\partial s_i} = -s_i \gamma_{ij} \frac{\partial F}{\partial s_j} + kT\gamma_{ij} \frac{\partial^2}{\partial s_i \partial s_j} F \tag{2.8}$$

We are interested in determining F near the saddle point y_i^0, and we therefore evaluate $\partial V / \partial y_i$ by using Eq. (2.2), and by defining

$$\varphi_{ij} = (M^{-1/2})_{il} V_{lm} (M^{-1/2})_{mj} \tag{2.9}$$

The matrix φ has one negative and $(n-1)$ positive eigenvalues (as does V). This important assertion is obvious if M is a multiple m $(m>0)$ of the unit matrix. In the general case, we observe that Eq. (2.9) implies

$$\det M \cdot \det \varphi = \det V \tag{2.10}$$

and that $(\det M)$ can be obtained by continuous deformation of $\det(m1) = m^n$, keeping all the eigenvalues positive. Under this operation, $\det\varphi$ obviously keeps its sign. Since $\det\varphi$ equals the product of the eigenvalues of φ, and since the latter depend continuously on the eigenvalues of M, the assertion follows.

Returning to Eq. (2.8) we require (as did Kramers) that F depends in the vicinity of the saddle point only on a single linear combination η of the variables $y_i - y_i^0$ and s_i,

$$F = F(\eta) \qquad \text{with} \quad \eta = a_i s_i - b_i(y_i - y_i^0) \tag{2.11}$$

Then, Eq. (2.8) takes the form

$$\left[-s_i b_i - (y_i - y_i^0)\varphi_{ij}a_j + s_i\gamma_{ij}a_j\right]\frac{\partial F}{\partial\eta} = kT(a_i\gamma_{ij}a_j)\frac{\partial^2 F}{\partial^2\eta} \tag{2.12}$$

This is consistent with the ansatz (2.11) only if the content of the square bracket is a multiple $(-H)$ of η. This condition, which must hold for any choice of s_i and $(y_i - y_i^0)$, implies that a_i and b_i obey the set of linear equations

$$\begin{aligned}(\gamma_{ij} + H\delta_{ij})a_j - b_i &= 0\\ \varphi_{ij}a_j + Hb_i &= 0\end{aligned} \tag{2.13}$$

These equations have a solution if H is a root of the equation

$$\det\begin{pmatrix}\gamma + H\cdot 1 & -1\\ \varphi & H\cdot 1\end{pmatrix} = 0 \tag{2.14a}$$

which can also be written in the form

$$\det(H^2\cdot 1 + H\gamma + \varphi) = 0 \tag{2.14b}$$

Given such a solution, Eq. (2.12) implies for F the form

$$F(\eta) = \left(\frac{G}{2\pi kT}\right)^{1/2}\cdot\int_{-\infty}^{\eta} d\xi \exp\left(-\frac{G\xi^2}{2kT}\right) \tag{2.15}$$

where

$$G = H\cdot(a_i\gamma_{ij}a_j)^{-1} \tag{2.16}$$

Equation (2.15) is meaningful only for $G>0$. This and the positive definiteness of γ imply $H>0$: H must be a *positive* root of Eq. (2.14). We now show that exactly one such root exists.

3. THE EIGENVALUE EQUATION (2.14)

For z real and $z \geqslant 0$, we consider the matrix

$$A(z) = z\gamma + \varphi \tag{3.1}$$

Since $A(z)$ is real and symmetric, there exists for each z an orthogonal transformation $O(z)$ which diagonalizes A. The eigenvalues $\sigma_i(z)$, $i = 1, \ldots, n$ of A increase monotonically with z. To show this we calculate

$$\frac{d}{dz}\sigma_i(z) = \frac{d}{dz}\left[O(z)A(z)O^T(z)\right]_{ii} = (O\gamma O^T)_{ii} \tag{3.2}$$

We have used $OO^T = 1$ and the definition (3.1). The assertion now follows since γ is positive definite:

$$\frac{d}{dz}\sigma_i(z) > 0 \qquad \text{for all } i \text{ and } z \geqslant 0 \tag{3.3}$$

For $z = 0$, the $\sigma_i(0)$ are equal to the eigenvalues φ_i of φ_{ij}, with $\varphi_1 < 0$ and $\varphi_l > 0$, $l \geqslant 2$. For $z \to \infty$, the σ_i tend towards the eigenvalues of $z \cdot \gamma$ which are all positive. This and relation (3.3) imply that $\sigma_l(z) > 0$ for $l \geqslant 2$ and $0 \leqslant z < \infty$, and that $\sigma_1(z)$ increases monotonically from its value $-|\varphi_1|$ at $z = 0$, intersecting the real z axis at some finite value of z.

Returning to Eq. (2.14b), we multiply both sides by $\det(O(H))$ and by $\det(O^T(H))$. This yields

$$\det\left(H^2\delta_{ij} + \sigma_i(H)\delta_{ij}\right) = \prod_{i=1}^{n}\left[H^2 + \sigma_i(H)\right] = 0 \tag{3.4}$$

For $H > 0$ we have $H^2 + \sigma_l(H) > 0$ if $l \geqslant 2$, while $H^2 + \sigma_1(H)$ rises monotonically with H from its value $-|\varphi_1|$ at $H = 0$, intersecting the real H axis at some finite value of H. This proves the assertion.

The construction just used can also be applied to determine the range of values of the positive root of Eqs. (2.14). Since σ_1 increases monotonically with increasing strength β_0 ($\beta_0 > 0$) of the matrix γ (or of the friction tensor β), the root of Eq. (2.14) decreases monotonically with β_0,

$$\frac{dH}{d\beta_0} < 0 \tag{3.5}$$

Moreover, $H = |\varphi_1|^{1/2}$ for $\beta_0 = 0$. (Note that this limit makes no physical sense and is used here only to obtain an upper limit on H.) This and relation (3.5) imply

$$0 \leqslant H \leqslant |\varphi_1|^{1/2} \tag{3.6}$$

4. THE DIFFUSION RATE

The Fokker–Planck equation (2.6) implies the continuity equation

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n\, W + \frac{\partial}{\partial y_i}\int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n \cdot s_i W = 0 \quad (4.1)$$

This shows that the current density, a vector in the n-dimensional space $\{y_1, \ldots, y_n\}$ is given by

$$j_i = \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n\, s_i W \quad (4.2)$$

The total diffusion current I is given by integrating $\mathbf{j}$ over the $(n-1)$-dimensional hyperplane S through the saddle point y_i^0 with normal vector in the direction of $\mathbf{j}$. We calculate $\mathbf{j}$ and I, using for W the quasistationary expressions (2.7) and (2.15).

Using

$$s_i \exp\left\{-\frac{1}{2kT}s_l s_l\right\} = -kT\frac{\partial}{\partial s_i}\exp\left(-\frac{1}{2kT}s_l s_l\right)$$

and a partial integration, we reduce the calculation of j_i in Eq. (4.2) to an n-dimensional Gaussian integral. This yields

$$j_i = a_i\left(\frac{GkT}{2\pi}\right)^{1/2} \cdot (2\pi kT)^{n/2}$$
$$\times (1 + Ga_l a_l)^{-1/2} \cdot \exp\left\{-\frac{1}{2kT}\cdot\frac{G}{1+Ga_l a_l}\cdot\left[(y_m - y_m^0)b_m\right]^2\right\}$$
$$\times \exp\left[-\frac{1}{2kT}(y_i - y_i^0)\varphi_{ij}(y_j - y_j^0)\right] \quad (4.3)$$

The eigenvalue equations (2.13) and the definition (2.16) of G imply

$$\frac{G}{1+Ga_l a_l} = \frac{H}{a_i\gamma_{ij}a_j + Ha_l a_l} = -\frac{H^2}{a_i\varphi_{ij}a_j} > 0 \quad (4.4)$$

where we have also used that γ is positive definite. Hence, using the second set of Eqs. (2.13) once again, we find

$$j_i = \frac{a_i H}{|a_l\varphi_{lk}a_k|^{1/2}}\cdot\frac{(2\pi kT)^{(n+1)/2}}{2\pi}\cdot\exp\left[-\frac{1}{2kT}(y_m - y_m^0)\tilde{\varphi}_{mn}(y_n - y_n^0)\right] \quad (4.5)$$

The real and symmetric matrix $\tilde{\varphi}$ is given by

$$\tilde{\varphi}_{mn} = \varphi_{mn} - \frac{\varphi_{mi}a_i a_j \varphi_{jn}}{a_k\varphi_{kl}a_l} \quad (4.6)$$

The fission current has the direction of the vector **a**. The inequality (4.4) and the second set of eigenvalue equations (2.13) imply that for $H > 0$, $(\mathbf{a} \cdot \mathbf{b}) > 0$. The definition of η in Eq. (2.12) then shows that if **y** has the direction of **j** (or **a**), and increases, η tends to $-\infty$. This is consistent with Eq. (2.15) and the boundary conditions imposed on F.

The total current I is obtained by integrating j_i over the $(n-1)$-dimensional hyperplane S through the point $\{y_k^0\}$ with normal vector in the direction of **j** or, equivalently, of **a**. We observe that $\tilde{\varphi}_{mn}a_n = 0$. This shows that the remaining $(n-1)$ eigenvectors of $\tilde{\varphi}$ (which are orthogonal upon **a**) span the hypersurface S. Denoting the associated eigenvalues by λ_i ($i = 2, \ldots, n$) and assuming that all $\lambda_i > 0$ (this is demonstrated below), we see that the calculation of I reduces to an $(n-1)$-fold Gaussian integral. Hence,

$$I = \int d\mathbf{s}\,\mathbf{j} = \frac{(a_i a_i)^{1/2}}{|a_l \varphi_{lk} a_k|^{1/2}} \cdot \frac{H}{2\pi} \cdot (2\pi kT)^n \left(\prod_{i=2}^{n} \lambda_i \right)^{-1/2} \tag{4.7}$$

The diffusion rate is obtained by dividing I by the normalization of W. The latter is evaluated via integration of W around the minimum at $x_i = 0$, where $F = 1$. Introducing again the variables s_i, y_i, using Eq. (2.1) and defining

$$\tilde{W}_{ij} = (M^{-1/2})_{im} W_{ml} (M^{-1/2})_{lj} \tag{4.8}$$

we find that

$$\int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\, W$$
$$\cong (2\pi kT)^n (\det \tilde{W})^{-1/2} \exp\left(\frac{V_B}{kT} \right) \tag{4.9}$$

Hence, the diffusion rate R is given by

$$R = \frac{H}{2\pi} \cdot \frac{(a_k a_k)^{1/2}}{|a_l \varphi_{lm} a_m|^{1/2}} \cdot \frac{(\det \tilde{W})^{1/2}}{\prod_{i=2}^{n} \lambda_i^{1/2}} \cdot \exp\left(-\frac{V_B}{kT} \right) \tag{4.10}$$

It remains to show that $\lambda_i > 0$, $i \geqslant 2$. In doing so, we shall simplify Eq. (4.10) further. We write the eigenvalue equation for the matrix $\tilde{\varphi}$ of Eq. (4.6) in a basis in which φ is diagonal, with eigenvalues $\varphi_l > 0$ for $l \geqslant 2$ and $\varphi_1 < 0$. In this basis, the components of **a** are denoted by α_i. We do *not* use the summation convention. Then

$$\varphi_i z_i - \varphi_i \alpha_i \frac{\left(\sum_m \alpha_m \varphi_m z_m \right)}{\sum_l \varphi_l \alpha_l^2} = \lambda z_i \tag{4.11}$$

Simple algebraic manipulations show that the eigenvalues of $\tilde{\varphi}$ are the roots of the equation

$$1 = \frac{1}{\sum_l \varphi_l \alpha_l^2} \sum_i \frac{\alpha_i^2 \varphi_i^2}{\varphi_i - \lambda} \tag{4.12}$$

The inequality (4.4) implies $\sum_l \varphi_l \alpha_l^2 < 0$. Moreover, $\lambda = 0$ is a solution of Eq. (4.12), with associated eigenvector $\mathbf{a}$. These statements and the fact that $\varphi_l > 0$ for $l \geqslant 2$, $\varphi_1 < 0$ can be used to show graphically in an elementary fashion that the remaining $(n-1)$ roots of Eq. (4.12) are all positive, with

$$\begin{aligned} &\varphi_i < \lambda_i < \varphi_{i+1}, \qquad 2 \leqslant i \leqslant n-1 \\ &\varphi_n < \lambda_n \end{aligned} \tag{4.13}$$

Having demonstrated the positivity of λ_l for $l \geqslant 2$, we observe that the product $\prod_{i=2}^n \lambda_i$ can be evaluated directly as follows. Let $\mathbf{c}_2, \mathbf{c}_3, \ldots, \mathbf{c}_n$ be the normalized eigenvectors of $\tilde{\varphi}$ belonging to the eigenvalues $\lambda_2, \ldots, \lambda_n$, respectively. Then, $\prod_{i=2}^n \lambda_i = \prod_{i=2}^n (\mathbf{c}_i \tilde{\varphi} \mathbf{c}_i)$. Moreover, the orthogonality of the $\mathbf{c}_i$ implies $(\mathbf{c}_i \tilde{\varphi} \mathbf{c}_j) = \delta_{ij}(\mathbf{c}_i \tilde{\varphi} \mathbf{c}_i)$. Hence,

$$\begin{aligned} \prod_{i=2}^n \lambda_i &= \det_{(n-1)} (\mathbf{c}_i \tilde{\varphi} \mathbf{c}_j) = \det_{(n-1)} \left[\mathbf{c}_i \varphi \mathbf{c}_j - \frac{(\mathbf{c}_i \varphi \mathbf{a})(\mathbf{c}_j \varphi \mathbf{a})}{(\mathbf{a} \varphi \mathbf{a})} \right] \\ &= \frac{1}{(\mathbf{a}\varphi\mathbf{a})} \det_{(n)} \begin{pmatrix} (\mathbf{a}\varphi\mathbf{a}) & (\mathbf{a}\varphi\mathbf{c}_i) \\ (\mathbf{a}\varphi\mathbf{c}_j) & (\mathbf{c}_i\varphi\mathbf{c}_j) \end{pmatrix} \end{aligned} \tag{4.14}$$

We have used the definition (4.6) and denoted the dimensionality of the matrix of which we take the determinant by a lower index. The last step results from straightforward algebra of determinants. Let us assume that $\mathbf{a}$ is normalized in unity,

$$\mathbf{a} \cdot \mathbf{a} = 1 \tag{4.15}$$

[The normalization is clearly arbitrary; cf. Eq. (4.10).] Then, the n vectors $(\mathbf{a}, \mathbf{c}_2, \ldots, \mathbf{c}_n)$ form an orthogonal matrix $\tilde{O}$, and we have

$$\det_{(n)} \begin{pmatrix} (\mathbf{a}\varphi\mathbf{a}) & (\mathbf{a}\varphi\mathbf{c}_i) \\ (\mathbf{c}_j\varphi\mathbf{a}) & (\mathbf{c}_j\varphi\mathbf{c}_i) \end{pmatrix} = \left(\det_{(n)} \tilde{O}\right)\left(\det_{(n)} \tilde{O}^T\right)\left(\det_{(n)} \varphi\right) \tag{4.16}$$

However, $\det_{(n)} \tilde{O} = 1$. Putting all this together, we have

$$\prod_{i=2}^n \lambda_i = \frac{\det \varphi}{(\mathbf{a}\varphi\mathbf{a})} = \frac{|\det \varphi|}{|\mathbf{a}\varphi\mathbf{a}|} \tag{4.17}$$

and the rate expression becomes with the help of Eq. (4.15)

$$R = \frac{H}{2\pi} \frac{(\det \tilde{W})^{1/2}}{|\det \varphi|^{1/2}} \cdot \exp\left(-\frac{V_B}{kT}\right) \tag{4.18}$$

Recalling the definitions (2.9) and (4.8), we can write this as

$$R = \frac{H}{2\pi}\left(\frac{\det W}{|\det V|}\right)^{1/2} \cdot \exp\left(-\frac{V_B}{kT}\right) \tag{4.19}$$

Although we have evaluated R in the frame of coordinates $y_1, \ldots, y_n$, $s_1, \ldots, s_n$ the result remains the same in the original frame $x_1, \ldots, x_n$, $u_1, \ldots, u_n$. This follows from the fact that

$$R = \frac{d}{dt}\ln\left(\int_v dy_1 \ldots dy_n \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n\, W\right) \tag{4.20}$$

where v is the volume of y space containing the point $y_i = 0$ and bounded by the hypersurface S. The coordinate transformation (2.5) leaves R invariant.

5. DISCUSSION AND SUMMARY

We have shown that the width for fission over an n-dimensional potential barrier has the form

$$\Gamma_f = \frac{\hbar}{2\pi} \cdot \exp\left(-\frac{V_B}{kT}\right) \cdot \left(\frac{\det W}{|\det V|}\right)^{1/2} \cdot H \tag{5.1}$$

Here, $V_B > 0$ is the height of the fission barrier, W_{ij} is the positive definite matrix which defines via Eq. (2.1) the parabolic approximation to the potential surface near the minimum at $x_i = 0$, and V_{ij} is the matrix which defines via Eq. (2.3) the parabolic approximation to the potential surface near the saddle point at x_i^0. The constant H is the only positive root of the equation

$$\det(H^2 M + H\beta + V) = 0 \tag{5.2}$$

Here, M is the mass tensor, β the positive definite friction tensor, and H obeys the inequalities

$$0 \leqslant H \leqslant |\varphi_1|^{1/2} \tag{5.3}$$

where $\varphi_1 < 0$ is the only negative eigenvalue of the matrix $\varphi = M^{-1/2} V M^{-1/2}$.

These results obviously generalize the Kramers result of Eqs. (1.1)–(1.5) in a very satisfactory fashion. The dynamical factor $(\det W/|\det V|)^{1/2}$ describes the geometry of the fission valley and is independent of the friction tensor. Qualitatively speaking $(\det W/|\det V|)^{1/2}$ is bigger (smaller) than the corresponding factor $(W/|V|)^{1/2}$ in one dimension if the fission valley gets wider (more narrow) as we approach the saddle point. This is seen by expressing both determinants in terms of the eigenvalues of the matrices W and V. The factor H, limited by the inequalities (5.3), depends on the specific values of the three matrices M, β, and V. In the context of the nuclear fission problem, a discussion of this dependence forms the subject of a separate paper.[7]

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NOTE ADDED IN PROOF

Dr. R. Landauer has kindly called our attention to articles by H. C. Brinkman, *Physica* **22**:149 (1956), by R. Landauer and J. A. Swanson, *Phys. Rev.* **121**:1668 (1961), and by J. S. Langer, *Phys. Rev. Lett.* **21**:973 (1968), and *Ann. Phys. (N.Y.)* **54**:258 (1969). These articles address the same problem in a different physical context, and the solutions given are related to our result. We are grateful to Dr. Landauer for this information.

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